Semiclassical lukewarm black holes

Jerzy Matyjasek and Katarzyna Zwierzchowska
Institute of Physics, Maria Curie- Skłodowska University
pl. Marii Curie- Skłodowskiej 1, 20-031 Lublin, Poland
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The perturbative solutions to the semiclassical Einstein field equations describing spherically-symmetric and static lukewarm black hole are constructed. The source term is composed of the (classical) stress-energy tensor of the electromagnetic field and the renormalized stress-energy tensor of the quantized massive scalar field in a large mass limit. We used two different parametrizations. In the first parametrization we calculated the zeroth-order solution. Subsequently, making use of the quantum part of the total stress-energy tensor constructed in the classical background we calculated the corrections to the metric potentials and the corrections to the horizons. This procedure can be thought of as switching the quantized field on and analyzing its influence on the classical background via the back-reaction. In the second parametrization we are looking for a self-consistent lukewarm solution from the very beginning. This requires knowledge of a generic tensor which depend functionally on the metric tensor. The transformation formulas relating the line element in both parametrizations are given.

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I. INTRODUCTION

One of the most characteristic features of the Riessner-Nordström- deSitter black holes is the simultaneous occurrence of the three horizon-like surfaces: the inner horizon, the event horizon and the cosmological horizon. This reflects the fact that the equation $g_{tt}(r) = 0$ has four real roots (not necessarily distinct), three of which are positive and represent horizons whereas the negative root has no physical interpretation. This leads to a number of special cases with the near-horizon geometries described by the Bertotti-Robinson [1, 2], Nariai [3, 4] or Plebanski-Hacyan [5] line elements. Among various allowable black hole configurations the class of solutions in which the temperature of the event horizon equals the temperature of the cosmological horizon is special [6–8]. It is because the mean values of the characteristics of the quantized fields, such as the field fluctuation and the renormalized stress-energy tensor are expected to be regular in the thermal state at the natural temperature. These expectations have been confirmed by a direct calculation of the stress-energy tensor in the two dimensional setting [9] and the field fluctuation of the massive quantized scalar field in the full four dimensional geometry [10]. The black holes for which both temperatures are equal are usually referred to as the lukewarm black holes. Recent analyses include [11–13]. It should be noted that there is no lukewarm configuration for the Schwarzschild-de Sitter black hole.

Since the stress-energy tensor of the quantized fields contribute to the source term of the semiclassical Einstein field equations, the resulting geometry changes due to the back reaction process. Consequently, the natural question arises: Whether or not it is possible to construct the lukewarm black hole in the semiclassical gravity. In Ref [12] a related problem has been considered in the context of the (classical) quadratic gravity. Specifically, it has been shown, that although the full, detailed answer is beyond our capabilities, it is possible to provide an affirmative answer to the restricted problem in which the differential equations describing the model are solved perturbatively. These results can also be viewed as a first step towards incorporation of the quantum effects into the picture. It is because the renormalized stress-energy tensor of the quantized massive field in the large mass limit may be approximated by the object constructed from the curvature tensor, its covariant derivatives and contractions. In this approach one ignores the particle creation which is a nonlocal phenomenon. In spite of this limitation this framework is still the most general one not restricted to any particular type of symmetry.

Here we generalize the results of Ref. [12] in a twofold way: First, we employ the renormalized stress-energy tensor of the massive scalar field with the arbitrary curvature coupling constructed within the framework of the Schwinger-DeWitt approximation to construct the semiclassical lukewarm black hole. Secondly, we study the relation between the results expressed in terms of the radii of the cosmological and event horizons of the classical lukewarm solution and the analogous results constructed in $(r_+, \tilde{r}_c)$ parametrization, where r_+ is the exact location of the event horizon of the semiclassical black hole and $\tilde{r}_c$ describes the cosmological horizon in a zeroth-order approximation.

A few words on the method are in order. First observe that first three terms of the Schwinger-DeWitt expansion are divergent and can be absorbed into the (classical) gravitational action with the cosmological constant and the quadratic terms [14]. This renormalizations leaves us with the two additional parameters, say, α and β which should be determined observationally. Their exact values are presently unknown, it is expected, however, that they are small, otherwise they would lead to various observational effects. To simplify calculations, in what follows, we shall set them

to zero. The second observation is related to the perturbative approach in the effective theories. In fact it may be the only method to deal with them. Indeed, since the semiclassical gravity involves six-order derivatives of the metric their nonperturbative solutions may appear to be spurious and one has to invent a method for systematic selecting physical ones. It seems that the acceptable solutions, when expanded in powers of the small parameter, should reduce to those obtained within the framework of perturbative approach. Finally observe that there are good reasons to believe that (in most cases) the black hole exists as perturbative solution of the higher order equations provided it exists classically [15].

II. CLASSICAL LUKEWARM BLACK HOLES

A very convenient representation of the line element describing the lukewarm black holes in the Einstein-Maxwell theory with the (positive) cosmological constant is that in terms of the horizons. Denoting the location of the event horizon by a and the location of the cosmological horizon by b the line element may be written in the form

$$ds^2 = -f_0(r)dt^2 + \frac{1}{f_0(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1}$$

where

$$f_0(r) = \left(1 - \frac{ab}{(a+b)r}\right)^2 - \frac{r^2}{(a+b)^2}. \tag{2}$$

Such a configuration is allowed provided

$$Q^2 = \left(\frac{ab}{a+b}\right)^2 \quad \text{and} \quad \Lambda = \frac{3}{(a+b)^2}. \tag{3}$$

It means that the electric charge Q (if there is a magnetic charge, P, Q^2 must be substituted by $Z^2 = Q^2 + P^2$) and Λ completely determines the lukewarm solution.

The simplest way to demonstrate that the line element of the lukewarm Reissner-Nordström-de Sitter black hole can be expressed solely in terms of a and b is to solve the Einstein-Maxwell equations with the cosmological term for a general static and spherically-symmetric metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4}$$

with the boundary conditions $B^{-1}(a) = 0$ and $A(b)B(b) = 1$. Solving the field equations and making use of the boundary conditions yields

$$A(r) = \frac{1}{B(r)} = 1 - \frac{a}{r} - \frac{Q^2}{ra} + \frac{\Lambda a^3}{3r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \tag{5}$$

which, with the substitution

$$M_H = \frac{a}{2} - \frac{Q^2}{2a} + \frac{\Lambda a^2}{6} \tag{6}$$

leads to the line element (4) written in a more familiar form

$$A(r) = B^{-1}(r) = 1 - \frac{2M_H}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}. \tag{7}$$

We shall refer to M_H as the horizon-defined mass. Simple analysis indicate that equation $A(r) = 0$ can have, depending on the values of the parameters, three, two or one distinct positive solutions. The above configurations can, therefore, have three distinct horizons located at zeros of $A(r)$, a degenerate and a nondegenerate horizon, and, finally, one triply degenerate horizon. Let us denote remaining solutions of the equation $A(r) = 0$ by r_{--} and r_- ($r_{--} < r_- \le a \le b$). Solving of the system of equations

$$A(b) = 0 \quad \text{and} \quad \kappa(a) + \kappa(b) = 0, \tag{8}$$

where

$$\kappa(r_i) = \frac{1}{2}\left(-g_{tt}g_{rr}\right)^{-1/2}\frac{d}{dr}g_{tt} \tag{9}$$

with respect to Λ and Q^2 one obtains (3). Finally, substituting the thus obtained Λ and Q^2 into Eq. 5 gives (1).

III. SEMICLASSICAL LUKEWARM BLACK HOLES

A. General equations in (a, b) parametrization

Now, let us consider the semiclassical Einstein field equations

$$R_i^j - \frac{1}{2} R \delta_i^j + \Lambda \delta_i^j = 8\pi \left(T_i^{(em)j} + T_i^{(q)j} \right), \tag{10}$$

where $T_i^{(em)j}$ is the stress-energy tensor of the electromagnetic field and $T_i^{(q)j}$ is the renormalized stress-energy tensor of some quantized test field or fields, evaluated in suitable state. Optimally, the renormalized stress-energy tensor of the quantized fields should functionally depend on the metric tensor, or, at least, on the wide class of geometries. Unfortunately, because of the mathematical complexity of the problem one is forced to refer either to the analytic approximations or numerics or both. Here we shall use the renormalized stress-energy tensor of the massive scalar field with an arbitrary curvature coupling in a large mass limit. Such a tensor can be calculated from the effective action, W_R, constructed within the framework of the Schwinger-DeWitt method [16]. Although the results appear to be state-independent the formulae used in this paper have been constructed with the assumption that the Green functions are defined in the spaces with the positive-definite metric. Of course, in this approach one ignores creation of real particles but the analyses that have been carried out so far suggest that for sufficiently massive fields it provides a reasonably accurate approximation [17].

The approximate renormalized one-loop effective action of the quantized massive fields in the large mass limit is given by a well-known formula

$$W_R = \frac{1}{32\pi^2} \sum_{n=3}^{\infty} \frac{(n-3)!}{(m^2)^{n-2}} \int d^4x \sqrt{g} [a_n], \tag{11}$$

where each $[a_n]$ has dimensionality of $[length]^{-2n}$ and is constructed from the Riemann tensor, its covariant derivatives up to $2n-2$ order and appropriate contractions. For the technical details of this approach the reader is referred, for example, to Refs. [18, 19] and the references cited therein. Inspection of Eq. (11) shows that the lowest term of the approximate W_R is to be constructed from the (integrated) coincidence limit of the fourth Hadamard-Minakshisundaram-DeWitt-Seely coefficient, $[a_3]$, whereas the next to leading term should be constructed form $[a_4]$. Here we will confine ourselves to the first term of the expansion (11) and briefly discuss some general features of the second-order term at the end of the paper. The approximate renormalized stress-energy tensor can be constructed from W_R using the standard relation

$$T^{ab} = \frac{2}{g^{1/2}} \frac{\delta}{\delta g_{ab}} W_R. \tag{12}$$

It should be noted that even if the renormalized stress-energy tensor is known the resulting semiclassical field equations are far too complicated to be solved exactly. Since the quantum effect are expected to be small, it is reasonable to assume that the quantum-corrected lukewarm black hole is described by a set of parameters that are close to their classical counterparts. Introducing slightly distorted metric potentials

$$A(r) = f_0(r) + \alpha(r) \tag{13}$$

and

$$1/B(r) = f_0(r) + \beta(r), \tag{14}$$

where $\alpha(r)$ and $\beta(r)$ are small corrections to the main approximation, expanding and retaining only the linear terms in the resulting differential equations one obtains

$$\frac{1}{r} \frac{d\beta(r)}{dr} + \frac{\beta(r)}{r^2} - \frac{1}{12\pi m^2} t_t^t = 0 \tag{15}$$

and

$$\frac{1}{r} \frac{d\alpha(r)}{dr} - 2p(r)\alpha(r) + q(r)\beta(r) - \frac{1}{12\pi m^2} t_r^r = 0, \tag{16}$$

where

$$p(r) = \frac{a^2br - a^2b^2 + ab^2r - r^4}{r^2\,(b-r)\,(a-r)\,(ab - ar - br - r^2)} \tag{17}$$

and

$$q(r) = \frac{a^2b^2 - r^2a^2 - 2\,r^2ab + 3\,r^4 - r^2b^2}{r^2\,(b-r)\,(r-a)\,(ab - ar - br - r^2)}. \tag{18}$$

The general solution to the system can be written

$$\beta(r) = \frac{1}{12\pi m^2 r}\int t_t^t(r) r^2 dr + C_1 \tag{19}$$

$$\alpha(r) = -\frac{P(r)}{6\pi m^2}\int \frac{1}{P(r)}\left(12q(r)\beta(r)\pi m^2 - t_r^r(r)\right)dr + P(r)C_2, \tag{20}$$

where

$$P(r) = \exp\left(\int p(r)dr\right). \tag{21}$$

and, consequently, the construction of the general solution reduces to two quadratures. Here $t_a^b = 96\pi^2 m^2 T_a^{(q)b}$. In deriving the above (formal) solution we have ignored subtleties associated with the stress-energy tensor itself, i.e., it is tacitly assumed that it is regular on the horizons. Fortunately, it turns out that the stress-energy tensor considered in this paper is free of such complications. The integration constants C_1 and C_2 should be determined from physically motivated boundary conditions.

B. The approximate stress-energy tensor

To solve the semiclassical Einstein field equations in a self-consistent way the renormalized stress-energy tensor describing the quantized source term is required. Such a tensor for a given field should functionally depend on the generic metric tensor. Unfortunately, because of the unavoidable complexities of the problem, its exact form is unknown. Provided we are interested in the analytical calculations all we can do is to look for some reasonable approximation [19–25].

The first-order approximation to the renormalized effective action of the quantized massive scalar field with arbitrary coupling to the curvature ξ satisfying the covariant Klein-Gordon equation

$$\left(\Box\, -\, \xi R\, -\, m^2\right)\phi\, =\, 0, \tag{22}$$

can be constructed from the coincidence limit of the coefficient

$$[a_3] = a_3^{(0)} + \xi a_3^{(1)} + \xi^2 a_3^{(2)} + \xi^3 a_3^{(3)}, \tag{23}$$

where

$$\begin{aligned} a_3^{(0)} \; = \; & \frac{11}{1680}R^3 + \frac{17}{5040}R_{;a}R^{;a} - \frac{1}{2520}R_{ab;c}R^{ab;c} - \frac{1}{1260}R_{ab;c}R^{ac;b} \\ + \; & \frac{1}{560}R_{abcd;e}R^{abcd;e} + \frac{1}{180}RR_{;a}{}^{a} + \frac{1}{280}R_{;a}{}^{a}{}_{b}{}^{b} + \frac{1}{420}R_{;ab}R^{ab} \\ - \; & \frac{1}{630}R_{ab;c}{}^{c}R^{ab} - \frac{109}{2520}RR_{ab}R^{ab} + \frac{73}{1890}R_{ab}R_{c}{}^{a}R^{bc} + \frac{1}{210}RR_{abcd}R^{abcd} \\ + \; & \frac{1}{105}R_{ab;cd}R^{acbd} + \frac{19}{630}R_{ab}R_{cd}R^{acbd} - \frac{1}{189}R_{abcd}R_{ef}{}^{ab}R^{cdef}, \end{aligned} \tag{24}$$

$$\begin{aligned} a_3^{(1)} \; = \; & -\frac{1}{72}R^3 - \frac{1}{30}R_{;a}R^{;a} - \frac{11}{180}RR_{;a}{}^{a} - \frac{1}{180}RR_{abcd}R^{abcd} \\ - \; & \frac{1}{60}R_{;a}{}^{a}{}_{b}{}^{b} - \frac{1}{90}R_{;ab}R^{ab} + \frac{1}{180}RR_{ab}R^{ab}, \end{aligned} \tag{25}$$

$$a_3^{(2)} = \frac{1}{12}R^3 + \frac{1}{12}R_{;a}R^{;a} + \frac{1}{6}RR_{;a}{}^{a}, \tag{26}$$

and

$$a_3^{(3)} = -\frac{1}{6}R^3 \tag{27}$$

It is evident that the approximate stress-energy tensor of the quantized massive field constructed from the effective action W_R depends functionally on the metric as it is constructed solely from the Riemann tensor, its contractions and covariant derivatives to some required order. The type of the field enters the general formulas by the spin-dependent numeric coefficients. There is no need, when computing the stress-energy tensor, to retain all terms in $[a_3]$. Indeed, the total divergences can be discarded and further simplifications of the effective action are possible. Here we display the coefficient $[a_3(x,x')]$ in its full form simply because it will be of use in the calculations of the field fluctuation.

Now, repeating the steps of Refs [21, 22] (to which the reader is referred for additional informations), after some algebra, one obtains the covariantly conserved tensor (12), with the tensor t_a^b is given by

$$\begin{aligned} t_t^t \;=\; & -\frac{2327a^6b^6}{105r^{12}(a+b)^6} + \frac{2452a^5b^5}{35r^{11}(a+b)^5} - \frac{8611a^4b^4}{105r^{10}(a+b)^4} + \frac{643a^4b^4}{35r^8(a+b)^6} + \frac{4442a^3b^3}{105r^9(a+b)^3} \\ & - \frac{2468a^3b^3}{105r^7(a+b)^5} - \frac{57a^2b^2}{7r^8(a+b)^2} + \frac{289a^2b^2}{35r^6(a+b)^4} + \frac{a^2b^2}{15r^4(a+b)^6} + \xi^3\left(\frac{432}{(a+b)^6} - \frac{432a^2b^2}{r^4(a+b)^6}\right) \\ & - \frac{37}{21(a+b)^6} + \xi^2\left(\frac{216a^2b^2}{r^4(a+b)^6} - \frac{216}{(a+b)^6}\right) + \xi\left(\frac{546a^6b^6}{5r^{12}(a+b)^6} - \frac{1808a^5b^5}{5r^{11}(a+b)^5} + \frac{6664a^4b^4}{15r^{10}(a+b)^4}\right. \\ & \left. - \frac{462a^4b^4}{5r^8(a+b)^6} - \frac{240a^3b^3}{r^9(a+b)^3} + \frac{656a^3b^3}{5r^7(a+b)^5} + \frac{48a^2b^2}{r^8(a+b)^2} - \frac{48a^2b^2}{r^6(a+b)^4} - \frac{26a^2b^2}{r^4(a+b)^6} + \frac{174}{5(a+b)^6}\right), \end{aligned} \tag{28}$$

$$\begin{aligned} t_r^r \;=\; & \frac{421a^6b^6}{105r^{12}(a+b)^6} - \frac{52a^5b^5}{3r^{11}(a+b)^5} + \frac{949a^4b^4}{35r^{10}(a+b)^4} - \frac{29a^4b^4}{3r^8(a+b)^6} - \frac{646a^3b^3}{35r^9(a+b)^3} - \frac{37}{21(a+b)^6} \\ & + \frac{1604a^3b^3}{105r^7(a+b)^5} + \frac{33a^2b^2}{7r^8(a+b)^2} - \frac{97a^2b^2}{15r^6(a+b)^4} + \xi^3\left(\frac{432}{(a+b)^6} - \frac{432a^2b^2}{r^4(a+b)^6}\right) + \frac{29a^2b^2}{15r^4(a+b)^6} \\ & + \xi^2\left(\frac{216a^2b^2}{r^4(a+b)^6} - \frac{216}{(a+b)^6}\right) + \xi\left(-\frac{78a^6b^6}{5r^{12}(a+b)^6} + \frac{336a^5b^5}{5r^{11}(a+b)^5} - \frac{1592a^4b^4}{15r^{10}(a+b)^4} + \frac{42a^4b^4}{r^8(a+b)^6}\right. \\ & \left. + \frac{368a^3b^3}{5r^9(a+b)^3} - \frac{336a^3b^3}{5r^7(a+b)^5} - \frac{96a^2b^2}{5r^8(a+b)^2} + \frac{144a^2b^2}{5r^6(a+b)^4} - \frac{178a^2b^2}{5r^4(a+b)^6} + \frac{174}{5(a+b)^6}\right) \end{aligned} \tag{29}$$

and

$$\begin{aligned} t_\theta^\theta \;=\; & -\frac{497a^6b^6}{15r^{12}(a+b)^6} + \frac{11404a^5b^5}{105r^{11}(a+b)^5} - \frac{13903a^4b^4}{105r^{10}(a+b)^4} + \frac{1769a^4b^4}{105r^8(a+b)^6} + \frac{2486a^3b^3}{35r^9(a+b)^3} - \frac{a^2b^2}{r^4(a+b)^6} \\ & - \frac{108a^3b^3}{5r^7(a+b)^5} - \frac{99a^2b^2}{7r^8(a+b)^2} + \frac{683a^2b^2}{105r^6(a+b)^4} - \frac{37}{21(a+b)^6} + \xi^3\left(\frac{432a^2b^2}{r^4(a+b)^6} + \frac{432}{(a+b)^6}\right) \\ & + \xi^2\left(-\frac{216a^2b^2}{r^4(a+b)^6} - \frac{216}{(a+b)^6}\right) + \xi\left(\frac{702a^6b^6}{5r^{12}(a+b)^6} - \frac{2272a^5b^5}{5r^{11}(a+b)^5} + \frac{8216a^4b^4}{15r^{10}(a+b)^4}\right. \\ & \left. - \frac{342a^4b^4}{5r^8(a+b)^6} - \frac{1456a^3b^3}{5r^9(a+b)^3} + \frac{416a^3b^3}{5r^7(a+b)^5} + \frac{288a^2b^2}{5r^8(a+b)^2} - \frac{24a^2b^2}{r^6(a+b)^4} + \frac{154a^2b^2}{5r^4(a+b)^6} + \frac{174}{5(a+b)^6}\right). \end{aligned} \tag{30}$$

These components can also be constructed from the Euler-Lagrange equations [25] with the Lagrangian depending on the time and radial components of the metric tensor, their derivatives and coordinate r. Because of the complexity of the calculations of the stress-energy tensor this alternative approach may serve as a useful check.

In order to establish the regularity of the stress-energy tensor at the horizon it is necessary to transform it into the coordinate system that is regular there. Alternatively, one can use in this regard a freely falling frame. For radial motion, the orthogonal vectors of the frame are the unit tangent to the geodesic $e^i_{(0)}$ and the three spacelike mutually perpendicular unit vectors $e^i_{(j)}$. Integrating the geodesic equation one obtains

$$u^a = \left[\frac{C}{A}, -\sqrt{\frac{1}{B}\left(\frac{C^2}{A}-1\right)}, 0, 0\right] \tag{31}$$

and

$$n^a = \left[\pm\frac{\sqrt{C^2 - A}}{A}, \mp\frac{C}{\sqrt{AB}}, 0, 0\right], \tag{32}$$

where C is the energy per unit mass along the geodesic. Elementary manipulations shows that the components $\tilde{T}_{(0)(0)}$, $\tilde{T}_{(1)(1)}$ and $\tilde{T}_{(0)(1)}$ of the stress-energy tensor in a freely falling frame are given by

$$\tilde{T}_{(0)(0)} = \frac{C^2}{A}\left(T_r^r - T_t^t\right) - T_r^r, \tag{33}$$

$$\tilde{T}_{(1)(1)} = \frac{C^2}{A}\left(T_r^r - T_t^t\right) + T_r^r \tag{34}$$

and

$$\tilde{T}_{(0)(1)} = \tilde{T}_{(1)(0)} = \frac{C\sqrt{C^2 - A}}{A}\left(T_r^r - T_t^t\right). \tag{35}$$

Since the difference between the radial and time components of the stress-energy tensor (28-30) factors as

$$T_r^{(q)r} - T_t^{(q)t} = A(r)F(r), \tag{36}$$

where $F(r)$ is a regular function, one concludes that the stress-energy tensor in a frame freely falling from the cosmological horizon or falling on the event horizon is regular in a physical sense.

C. Semiclassical lukewarm black holes in (a, b) parametrization

As is seen from Eqs. (19) and (20) the general solution of the linearized system (10) requires two simple quadratures, which, after some calculations, yield rather long expressions. The corrections to the "metric potentials" $1/B(r)$ and $A(r)$, i.e., the functions $\beta(r)$ and $\alpha(r)$ have the structure

$$\beta(r) = \frac{1}{\pi m^2 (a+b)^6}\sum_{i=0}^{3}\beta_i(r)\xi^i + \frac{C_1}{r} \tag{37}$$

and

$$\alpha(r) = \beta(r) + \frac{f_0(r)}{\pi m^2 (a+b)^4}\left[\sum_{i=0}^{1}\alpha_i(r)\xi^i - 6\xi^2 + 12\xi^3\right] - (a+b)^2 C_2, \tag{38}$$

where $f_0(r)$ is given by (1) and the exact form of the functions $\alpha_i(r)$ and $\beta_i(r)$ are relegated to the appendix. Of the two integration constants, only C_1 affects location of the horizons; the second integration constant is left free throughout the calculation. If it exists, the quantum-corrected lukewarm black hole must satisfy the same requirements as its classical counterpart, and, consequently, in order to determine the line element describing such a configuration, one has to solve the system of algebraic equations:

$$A(r_+) = A(r_c) = 0 \tag{39}$$

and

$$\kappa(r_+) + \kappa(r_c) = 0, \tag{40}$$

where $r_+ = a + r_+^{(1)}$ and $r_c = b + r_c^{(1)}$, with respect to C_1, $r_+^{(1)}$ and $r_c^{(1)}$. Here r_+ and r_c correspond respectively to the event horizon and the cosmological horizon, whereas $r_+^{(1)}$ and $r_c^{(1)}$ are small corrections. Now, simple manipulations give

$$r_+^{(1)} = \frac{1}{\pi m^2 (a-b)(a+b)^5}\left(W_+^{(0)} + W_+^{(1)}\xi\right) + \frac{1}{\pi m^2 (a-b)(a+b)^4}\left(W_+^{(2)}\xi^2 + W_+^{(3)}\xi^3\right) \tag{41}$$

and

$$C_1 = \frac{1}{\pi m^2 (a+b)^7}\left(V^{(0)} + V^{(1)}\xi\right) + \frac{1}{\pi m^2 (a+b)^5}\left(V^{(2)}\xi^2 + V^{(3)}\xi^3\right) \tag{42}$$

where functions $W_+^{(k)}$ and $V^{(k)}$ are listed in Appendix. It should be noted that

$$r_c^{(1)}(a,b) = -r_+^{(1)}(a \to -b, b \to -a). \tag{43}$$

i.e., the correction to the cosmological horizon equals minus the correction to the event horizon with simultaneous substitution $a \to -b$ and $b \to -a$. The integration constant C_2 can be determined using, for example, the condition $A(r_c)B(r_c) = 1$.

The equations (37) with (42) and (38) solve the problem completely: the semiclassical lukewarm configuration is characterized by the cosmological constant and (total) charge as given by (3), whereas the quantum corrections to the event and cosmological horizon are given by (41) and (43), respectively.

D. Semiclassical black holes in $(r_+, \tilde{r}_c)$ parametrization

Now we shall show that the above procedure is equivalent to a more familiar approach, in which one looks for a lukewarm solution parametrized by the exact location of the event horizon and the zeroth-order approximation to the cosmological horizon. First, let us assume that the cosmological constant is a parameter in a space of theories rather than the space of solutions. Now, one can solve the semiclassical Einstein field equations for a line element of the form (4) with

$$A(r) = \left(1 - \frac{2M(r)}{r}\right) e^{2\psi(r)} \tag{44}$$

and

$$B(r) = \left(1 - \frac{2M(r)}{r}\right)^{-1}, \tag{45}$$

where $M(r) = M_0(r) + M_1(r)$ and $\psi(r) = \psi_0(r) + \psi_1(r)$ and the functions $M_1(r)$ and $\psi_1(r)$ are small corrections to the main approximation. In constructing the linearized solution we adopt the natural conditions $M_0(r_+) = r_+/2$, $M_1(r_+) = 0$ and $\psi_0(r) = 0$, leaving unspecified the integration constant, say $\tilde{C}_2$, which appears as a result of integration of the differential equation for $\psi_1(r)$. The zeroth-order solution is therefore parametrized by the exact location of the event horizon, r_+, and the charge Q^2. The lukewarm configuration can be expressed in terms of r_+ and the zeroth-order approximation to the cosmological horizon $\tilde{r}_c$

$$A(r) = B^{-1}(r) = \left(1 - \frac{r_+ \tilde{r}_c}{(r_+ + \tilde{r}_c)\, r}\right)^2 - \frac{r^2}{(r_+ + \tilde{r}_c)^2}. \tag{46}$$

In this approach we do not attribute any physical significance to the zeroth-order solution. Once again, the first-order corrections can easily be constructed by the two simple quadratures and the quantum-corrected lukewarm black hole is characterized by the exact location of the event horizon, r_+, the location of the cosmological horizon

$$r_c = \tilde{r}_c + \delta(r_+, \tilde{r}_c) \tag{47}$$

and the relation between the (total) charge and r_+ and $\tilde{r}_c$:

$$Q^2 = \left(\frac{r_+ \tilde{r}_c}{r_+ + \tilde{r}_c}\right)^2 + \Delta(r_+, \tilde{r}_c) \tag{48}$$

By the assumption we made about the cosmological constant it is still given by

$$\Lambda = \frac{3}{(r_+ + \tilde{r}_c)^2}. \tag{49}$$

Although we have calculated both $\delta(r_+, \tilde{r}_c)$ and $\Delta(r_+, \tilde{r}_c)$ we shall not display them here, simply because they are rather lengthy. Moreover, they can easily be constructed form the formulas listed in the appendix by switching from (a, b) representation to $(r_+, \tilde{r}_c)$, (i. e., by the reverse procedure to that discussed below).

For the same black hole configuration one can switch from $(r_+, \tilde{r}_c)$ representation to (a, b) making use of the equations

$$\frac{3}{(r_+ + \tilde{r}_c)^2} = \frac{3}{(a+b)^2} \tag{50}$$

and

$$\left(\frac{r_+\tilde{r}_c}{r_+ + \tilde{r}_c}\right)^2 + \Delta(r_+, \tilde{r}_c) = \left(\frac{ab}{a+b}\right)^2 \tag{51}$$

Indeed, substituting

$$r_+ = a + a_1 \quad \text{and} \quad \tilde{r}_c = b + b_1, \tag{52}$$

where a_1 and b_1 are small corrections, into the system (50) and (51) one obtains

$$a_1 = -b_1 = \frac{(a+b)^2}{2ab(a-b)}\Delta(a, b). \tag{53}$$

Since we are interested in the first-order calculations, we can safely change the arguments of the function $\Delta(r_+, \tilde{r}_c)$. It can be demonstrated that r_+ and $r_c = b + b_1 + \delta(a, b)$ are given respectively by by (41) and (43), as expected. Moreover, both approaches yield identical results for the metric tensor, provided the integration constant $\tilde{C}_2$ is related to the constant C_2 by

$$\tilde{C}_2 = -(a+b)^2 C_2 - \frac{1}{\pi m^2 (a+b)^4}\left(\frac{37}{756} - \frac{29}{30}\xi + 6\xi^2 - 12\xi^3\right). \tag{54}$$

As before, to determine C_2 (and hence $\tilde{C}_2$) additional informations are required.

IV. FINAL REMARKS

In this paper we have constructed perturbative solutions to the semiclassical Einstein field equations describing spherically-symmetric and static lukewarm black hole. The total source term is composed of two parts: the (classical) stress-energy tensor of the electromagnetic field and the renormalized stress-energy tensor of the quantized massive scalar field in a large mass limit. In the course of our calculations we used two different parametrizations, and, assuming that the first-order results describe the same black hole configuration we constructed the transformation rules from the one parametrization to the other. In the parametrization (a, b) we first calculated the zeroth-order solution. Subsequently, making use of the quantum part of the total stress-energy tensor constructed in the classical background we calculated the corrections to the metric potentials and the corrections to the horizons. This procedure can be thought of as switching the quantized field on and analyzing its influence on the classical background via the back-reaction. On the other hand, in the parametrization $(r_+, \tilde{r}_c)$, we are looking for a self-consistent solution from the very beginning. This requires a generic tensor which depend functionally on the metric tensor. Since the calculations in that case are rather involved and produce complex results, we discussed them only briefly.

We conclude this paper with a number of comments:

1. Here we have considered only the quantized massive scalar fields with $\xi R\phi$ coupling in a large mass limit. Since the approximate stress energy tensors of the massive spinor and the massive vector fields in for a generic metric are known, the results presented here can easily be extended to these cases.

2. Once the renormalized stress-energy tensor is known a similar analysis can be carried out for the quantized massless fields. Unfortunately, although the renormalized stress-energy tensor of various fields are well documented in the Schwarzschild spacetime (see for example Refs. [26–35] and the references cited therein) less is known about T_a^b in more complicated geometries. The remarks made in subsection III B remain intact and it is crucial to check the regularity of the stress-energy tensor on the horizons.

3. Since the stress-energy tensor is constructed solely form the Riemann tensor, its derivatives and the metric one expects that similar calculations can be performed in other theories in which the higher curvature terms appear. For example, although we have considered only the main approximation to the stress-energy tensor constructed from the integrated coincidence limit of the Hadamard-DeWitt coefficient $a_3(x, x')$ the calculations can be extended, at the price of the technical complications, to the next-to-leading term involving functional derivatives of the coincidence limit of the coefficient $a_4(x, x')$. Preliminary calculations indicate that it is possible to construct the lukewarm black hole in such a case.

4. It could be shown that the approximation to the mean value of the field fluctuation in a large mass limit is given by [36]

$$\langle\phi^2\rangle = \frac{1}{16\pi^2}\sum_{n=2}^{N}\frac{(n-2)!}{m^{2(n-1)}}[a_n], \tag{55}$$

where $N-1$ is the number of terms retained in the expansion. Taking

$$[a_2] = \frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box R + \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 R^2 - \frac{1}{180}R_{ab}R^{ab} + \frac{1}{180}R_{abcd}R^{abcd} \tag{56}$$

and $[a_3]$ as given by (23-27) one can calculate the first two terms of the approximation. For example, routine calculations carried out in the spacetime of the Einstein-Maxwell lukewarm black hole (1) give the for the main approximation

$$16\pi^2 m^2\langle\phi^2\rangle = \frac{4}{15}\frac{a^2b^2}{r^6w^2} - \frac{8}{15}\frac{a^3b^3}{r^7w^3} + \frac{13}{45}\frac{a^4b^4}{r^8w^4} + \frac{29}{15w^4} - \frac{24}{w^4}\xi + \frac{72}{w^4}\xi^2, \tag{57}$$

where $w = a+b$.

5. Since the cosmological "constant" is expected to vary in time (see for example [37–39] and the references cited therein) it would be interesting to analyze the charged black holes in such a dynamic environment. This, however, would require a deeper understanding of quantum phenomena taking place in the spacetime of nonstatic black holes.

6. Extension of the results presented in this paper to the black hole spacetimes of $d-$dimensions requires detailed knowledge of the higher HMDS coefficients.

Some of the listed problems are under active investigations and the results will be published elsewhere.

Appendix

The geometry of the semiclassical lukewarm black hole in the (a,b) parametrization is described by the line element (4) with (13) and (14). The $\beta(r)$ is given by

$$\beta(r) = \frac{1}{\pi m^2(a+b)^6}\sum_{i=0}^{3}\beta_i(r)\xi^i + \frac{C_1}{r} \tag{A.1}$$

where

$$\begin{aligned}\beta_0(r) \;=\;& \frac{2327a^6b^6}{11340r^{10}} - \frac{613a^6b^5}{840r^9} + \frac{8611a^6b^4}{8820r^8} - \frac{2221a^6b^3}{3780r^7} + \frac{19a^6b^2}{140r^6} - \frac{613a^5b^6}{840r^9} + \frac{8611a^5b^5}{4410r^8} - \frac{2221a^5b^4}{1260r^7} + \frac{19a^5b^3}{35r^6}\\ &+\frac{8611a^4b^6}{8820r^8} - \frac{2221a^4b^5}{1260r^7} + \frac{1067a^4b^4}{2100r^6} + \frac{617a^4b^3}{1260r^5} - \frac{289a^4b^2}{1260r^4} - \frac{2221a^3b^6}{3780r^7} + \frac{19a^3b^5}{35r^6} + \frac{617a^3b^4}{1260r^5} - \frac{289a^3b^3}{630r^4}\\ &+\frac{19a^2b^6}{140r^6} - \frac{289a^2b^4}{1260r^4} - \frac{a^2b^2}{180r^2} - \frac{37r^2}{756},\end{aligned} \tag{A.2}$$

$$\begin{aligned}\beta_1(r) \;=\;& -\frac{91a^6b^6}{90r^{10}} + \frac{113a^6b^5}{30r^9} - \frac{238a^6b^4}{45r^8} + \frac{10a^6b^3}{3r^7} - \frac{4a^6b^2}{5r^6} + \frac{113a^5b^6}{30r^9} - \frac{476a^5b^5}{45r^8} + \frac{10a^5b^4}{r^7} - \frac{16a^5b^3}{5r^6}\\ &-\frac{238a^4b^6}{45r^8} + \frac{10a^4b^5}{r^7} - \frac{163a^4b^4}{50r^6} - \frac{41a^4b^3}{15r^5} + \frac{4a^4b^2}{3r^4} + \frac{10a^3b^6}{3r^7} - \frac{16a^3b^5}{5r^6} - \frac{41a^3b^4}{15r^5} + \frac{8a^3b^3}{3r^4}\\ &-\frac{4a^2b^6}{5r^6} + \frac{4a^2b^4}{3r^4} + \frac{13a^2b^2}{6r^2} + \frac{29r^2}{30}\end{aligned} \tag{A.3}$$

and

$$\beta_2(r) = -\frac{\beta_3(r)}{2} = -\frac{18a^2b^2}{r^2} - 6r^2. \tag{A.4}$$

The function $\alpha(r)$ is given by

$$\alpha(r) = \beta(r) + \frac{f_0(r)}{\pi m^2 (a+b)^4}\left[\sum_{i=0}^{1} \alpha_i(r)\xi^i - 6\xi^2 + 12\xi^3\right] - (a+b)^2 C_2, \tag{A.5}$$

where

$$\alpha_0(r) = -\frac{229a^4b^4}{840r^8} + \frac{184a^4b^3}{441r^7} - \frac{5a^4b^2}{28r^6} + \frac{184a^3b^4}{441r^7} - \frac{5a^3b^3}{14r^6} - \frac{5a^2b^4}{28r^6} + \frac{7a^2b^2}{180r^4} - \frac{37}{756} \tag{A.6}$$

and

$$\alpha_1(r) = \frac{13a^4b^4}{10r^8} - \frac{32a^4b^3}{15r^7} + \frac{14a^4b^2}{15r^6} - \frac{32a^3b^4}{15r^7} + \frac{28a^3b^3}{15r^6} + \frac{14a^2b^4}{15r^6} - \frac{a^2b^2}{5r^4} + \frac{29}{30}. \tag{A.7}$$

The location of the event horizon of the quantum-corrected lukewarm black hole is given by (41) with

$$W_+^{(0)} = \frac{17a^6}{317520b^3} + \frac{17a^5}{31752b^2} + \frac{41a^4}{4900b} + \frac{b^6}{504a^3} + \frac{2641a^3}{396900} + \frac{31b^5}{35280a^2} + \frac{719a^2b}{11340} + \frac{173b^4}{11340a} + \frac{1208ab^2}{11025} + \frac{89b^3}{2835}, \tag{A.8}$$

$$W_+^{(1)} = -\frac{a^4}{50b} - \frac{b^6}{180a^3} + \frac{109a^3}{300} - a^2b - \frac{11b^4}{180a} - \frac{473ab^2}{300} - \frac{b^3}{10} \tag{A.9}$$

and

$$W_+^2 = -\frac{W_+^{(3)}}{2} = -3a^2 + 9ab. \tag{A.10}$$

The integration constant C_1 is given by (42) with

$$\begin{aligned} V^{(0)} \;=\; & \frac{17a^7}{158760b^3} + \frac{17a^6}{15876b^2} + \frac{41a^5}{2450b} + \frac{24707a^4}{396900} + \frac{17b^7}{158760a^3} + \frac{1993a^3b}{11340} \\ & + \frac{17b^6}{15876a^2} + \frac{14039a^2b^2}{44100} + \frac{41b^5}{2450a} + \frac{1993ab^3}{11340} + \frac{24707b^4}{396900}, \end{aligned} \tag{A.11}$$

$$V^{(1)} = -\frac{a^5}{25b} - \frac{6a^4}{25} - \frac{89a^3b}{30} - \frac{439a^2b^2}{75} - \frac{b^5}{25a} - \frac{89ab^3}{30} - \frac{6b^4}{25} \tag{A.12}$$

and

$$V^{(2)} = -\frac{V^{(3)}}{2} = 18ab. \tag{A.13}$$

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